The Triangle Inequality

The triangle inequality is used in many proofs involving limits and their properties. For any two real numbers \( a \) and \( b \), the triangle inequality states that:

\[
|a + b| \leq |a| + |b|
\]

This is relatively easy to demonstrate by noting that for any two real numbers \( a \) and \( b \):

\[
ab \leq |ab| = |a||b|
\]

Starting with this fact, we note:

\[
\begin{align*}
ab & \leq |a||b| \\
2ab & \leq 2|a||b| \\
a^2 + 2ab & \leq a^2 + 2|a||b| \\
a^2 + 2ab + b^2 & \leq a^2 + 2|a||b| + b^2 \\
a^2 + 2ab + b^2 & \leq a^2 + 2|a||b| + b^2 = |a|^2 + 2|a||b| + |b|^2 \\
a^2 + 2ab + b^2 & \leq |a|^2 + 2|a||b| + |b|^2 \\
(a + b)^2 & \leq (|a| + |b|)^2 \\
|a + b|^2 & = (a + b)^2 \leq (|a| + |b|)^2 \\
|a + b|^2 & \leq (|a| + |b|)^2 \\
|a + b| & \leq |a| + |b|
\end{align*}
\]

which proves the triangle inequality.
Property: The limit of the sum equals the sum of the limits.

We can use the triangle inequality to prove various properties of limits. For example suppose \( f(x) \) and \( g(x) \) are two continuous functions and that \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a} g(x) \) both exist. Then:

\[
\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)
\]

That is, the limit of the sum is the sum of the limits. To show this let \( \lim_{x \to a} f(x) = F \) and \( \lim_{x \to a} g(x) = G \). We know \( F \) and \( G \) exist by hypothesis. By definition, for every \( \varepsilon > 0 \) there exists \( \delta_1 > 0 \) and \( \delta_2 > 0 \), such that whenever \( |x - a| < \delta_1 \) then \( |f(x) - F| < \frac{\varepsilon}{2} \) and whenever \( |x - a| < \delta_2 \) then \( |g(x) - G| < \frac{\varepsilon}{2} \).

Therefore:

\[
\varepsilon = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} > |f(x) - F| + |g(x) - G|
\]

whenever \( |x - a| < \delta = \min(\delta_1, \delta_2) \). Using the triangle inequality we have:

\[
\varepsilon > |f(x) - F| + |g(x) - G| \geq |f(x) - F + g(x) - G| = |f(x) + g(x) - (F + G)|
\]

Or, more specifically, for every \( \varepsilon > 0 \), there exist \( \delta > 0 \) such that whenever \( |x - a| < \delta \),

\[
|f(x) + g(x) - (F + G)| < \varepsilon .
\]

Therefore by definition, we have:

\[
\lim_{x \to a} (f(x) + g(x)) = F + G = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)
\]

Which proves the property that the limit of the sum is the sum of the limits.