Hints/Comments on AMATYC Student Math League Exam Fall 2007

1. If a can holds x ounces, then $x + \frac{13}{3}x = 64\ldots.$

2. If $a\Delta b = ab + b = (a+1)b$, then $(3\Delta 2) = (3+1)2 = 8$, $(2\Delta 3) = (2+1)3 = 9$, and so $(3\Delta 2)\Delta(2\Delta 3) = 8\Delta 9\ldots$

3. If the rectangle has dimensions x by y, then the entire design has area $(x+y)^2$, and the area of the interior square is $(x-y)^2$. So $(x-y)^2 = \frac{4}{9}(x+y)^2$, and we need to find $\frac{\text{length of rectangle}}{\text{side of central square}} = \frac{x}{x-y}$. Taking positive square roots on both sides of the first equation, we get $x - y = \frac{2}{3}(x+y)$, so $\frac{1}{3}x = \frac{5}{3}y$, so $x = 5y$...

4. If there are reports every 10 minutes, then there are 6 reports every hour, and so $\frac{6\text{ reports}}{1\text{ hour}} \cdot \frac{24\text{ hour}}{1\text{ day}} \cdot \frac{7\text{ day}}{\text{week}} = 1008\frac{\text{reports}}{\text{week}}\ldots.$.

5. If d is the number of dimes, n the number of nickels, then $d + n = 24$, and their total worth is $10d + 5n$ cents….

6. The area of $TUVW$ is the difference between the area of the large square with side 10 and the total area of the 4 right triangles with legs 2 and 8….

7. $s\,\frac{\text{ft}}{\text{min}} = s\frac{\text{ft}}{\text{min}} \cdot \frac{12\text{ in}}{1\text{ ft}} \cdot \frac{1\text{ min}}{60\text{ sec}} = \frac{s}{5}\frac{\text{in}}{\text{sec}}$, so $\frac{s}{5} = s - 16\ldots.$

8. $\frac{A+2B}{2} = 7$ and $\frac{A+2C}{2} = 8 \Rightarrow A + 2B = 14$ and $A + 2C = 16$. Adding the last two equations, $2A + 2B + 2C = 30$, so….

9. The largest allowable perfect square with 6 digits should have the form 9M9TYC….

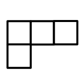
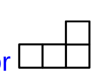
10. The greatest distance is the same as the main diagonal of a box that is 7 ft $\times$ 4 ft $\times$ 4ft….

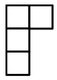
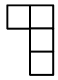
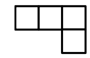
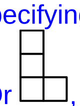
11. If F is the number of females, M the number of males, then $\frac{F}{F+M} = .40$. If 3 males are replaced by females, the total number does not change and $\frac{F+3}{F+M} = .44$. Subtracting the first equation, $\frac{3}{F+M} = .04$, or $3 = .04(F+M) \Rightarrow F + M = 75 \Rightarrow F = 25$ and $M = 50$, so…

12. There are two ways to assign the saxophone parts. For each of those possibilities, there are 3 ways to assign the trumpet parts, and for each of those….

13. $N + (2N)^2 = N(1 + 4N)$. This is factored, so it is not prime unless $N = 1$ or $N = -1\ldots$

14. The first 5 perfect squares are 1, 4, 9, 16, 25. 1 cannot be the sum of two positive integers, so the only possible sums are 4, 9, 16, 25. If we list each of the integers from 1 to 16 along with their possible partners, we see that 8 must be paired with 1, and we can determine all the other pairings….

15. The upper left corner could be filled by  or , each of which specifies the top half and allows two possibilities for the bottom half. Or the upper left corner could be filled by

 or , each specifying the left half and allowing two more possibilities. Or , allowing only one. Or , allowing only one.....

16. If you walk along the perimeter of the polygon, at each vertex you turn by an amount of an exterior angle. Thus for a regular n-gon, you make n turns of $\frac{360°}{n}$. So the possible integer exterior angles must divide 360....

17. We need to count how many fractions of the form $\frac{7m+5n}{3\cdot5\cdot7}$ are not in simplest terms when $\frac{m}{15}$ and $\frac{n}{21}$ are positive proper fractions in simplest terms. This means that $7m+5n$ is divisible by 3. (It won't be divisible by 5 or 7 because then either $\frac{m}{15}$ or $\frac{n}{21}$ could be reduced.) This implies that $m \equiv n(\text{mod } 3)$. For each of the four possibilities of m that are 1 mod 3, there are six possibilities for n, and for each of the four possibilities of m that are 2 mod 3, there are six possibilities for n....

18. If the two equations are added, $r(1+s)+t(1+s)=27$, or $(r+t)(1+s)=27 \Rightarrow (1+s)$ divides evenly into 27, so $1+s$ must be 1, 3, 9, or 27...

19. $\frac{1^2+2^2+3^2+\cdots+17^2}{17}=105$ and $\frac{2^2+3^2+\cdots+17^2+18^2}{17}=124$. The difference between these two is 19, and the two perfect squares differing by 19 are 81 and 100 $\Rightarrow k=24$...

20. $\angle L = 180° - (\angle S + 2\angle S) = 180° - 3\angle S$. By the Law of Sines, $\frac{7}{\sin(180°-3\angle S)} = \frac{9}{\sin\angle S}$ or $\frac{7}{\sin 3\angle S} = \frac{9}{\sin\angle S}$ $\Rightarrow$

$$\begin{aligned}
\frac{7}{9} = \frac{\sin 3\angle S}{\sin\angle S} &= \frac{\sin(\angle S + 2\angle S)}{\sin\angle S} \\
&= \frac{\sin\angle S\cos 2\angle S + \cos\angle S\sin 2\angle S}{\sin\angle S} \\
&= \frac{\sin\angle S\cos 2\angle S + \cos\angle S(2\sin\angle S\cos\angle S)}{\sin\angle S} \\
&= \cos 2\angle S + 2\cos^2\angle S \\
&= 4\cos^2\angle S - 1
\end{aligned}$$

so $\frac{16}{9} = 4\cos^2\angle S$. From the Law of cosines,

$$\begin{aligned}
(SL)^2 &= 7^2 + 9^2 - 2\cdot7\cdot9\cdot\cos 2\angle S \\
&= 130 - 14\cdot9\left(2\cos^2\angle S - 1\right) \\
&= 130 - 14\cdot9\left(\frac{8}{9} - 1\right) \\
&= 144
\end{aligned}$$

#4: The statement is sloppy. The "difference" depends on whether we consider number of reports *per year* (as intended) or in some other time interval.

#17: The statement is sloppy. Whether or not the we find sum of two fractions in simplest terms can depend on whether or not we use the LCD.