Hints/Comments on AMATYC Student Math League Exam Fall 2008

1. If $M \perp L$, then the slope of M is the negative reciprocal of the slope of L, so the slope of M is $\frac{-1}{2}$, and M has equation $y = \frac{-1}{2}x + 3$. Check to see which points satisfy this equation....

2. If S is Sue's old salary ($), and L is Lisa's salary, then $1.05S = L + 1200$ and $1.01S = L$...

3. $ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = a(x+1)(x-r)$, ...

4. There are two ways of having 45¢ with 9 coins if 5 of them are pennies, but there is only one way without pennies...

5. If $(a, b) = (kx, ky)$ for integers $k, x,$ and y, then (a, b) is hidden because of (x, y). Thus the visible points have coordinates that are relatively prime...

6. After 4 jumps the flea progresses $1 + 2 + 3 + 2 = 8$, so that after 20 jumps it's $40 \equiv 4 \pmod{12}$, after 100 it's 8 mod 12, after 1000 it's 8 mod 12, so it's 4 mod 12 after 2000 jumps, and .....

7. The midpoints of any quadrilateral form vertices of a parallelogram....

8. There are 15 subsets to check....

9. $x = \frac{b+c}{a} = \frac{a+c}{b} \Rightarrow b^2 + bc = a^2 + ac \Rightarrow b^2 - a^2 + c(b-a) = 0$...

10. $f(x) = \frac{(x-2)(x-20)}{(x-2)(x+15)}$.....

11. The largest allowable number should have the form 9897YC, and in order to be divisible by 55, the C must be 0 or 5, and to be divisible by 11, $9 - 8 + 9 - 7 + Y - C$ must be divisible by 11...

12. $3^6 = 729$, 5^6 is too large. $2009 = 3^6 + 2^{10} + 2^8$....

13. Because $365 = 1 + 7 \cdot 52$, Feb 1 will move 2 days forward to Sunday (after a leap year) and one day forward otherwise $\Rightarrow$ it moves forward 5 days every 4 years, and it won't be a multiple of 7 days until 28 years. So in 2036, February will have 5 Fridays, but having "moved forward" 35 days or 7 weeks (an odd number of weeks), the first Friday in 2036 is not a payday. So after waiting another 28 years, in 2064, there will be 5 Fridays, and the first, third, and fifth will be paydays.

14. E and B are both telling the truth or both lying, but D and A are opposite. Since only 2 are lying, E and B must be telling the truth. The third truth teller cannot be A because A says D guilty $\Rightarrow$ C also tells the truth (too many telling the truth). So A lies $\Rightarrow$ D is telling the truth and C lies $\Rightarrow$ ....

15. The product of two linear sequences would be quadratic. The first difference are -6 and -78, and because the second differences should be constant, the next first difference

should be -150, so the next term should be $384 - 150 = 234$. None of the answers shown is correct.

16. If you draw the angle bisectors of the three angles, they meet at the center of the circumscribed circle. Name the radius of that circle r, name half the angles at A, B, and C respectively α, β, and γ, name the lengths from A to the nearby points of tangency x, name the tangents from B y, and from C z. Then the expression to evaluate is

$$\begin{aligned}\frac{\tan(\alpha+\beta)}{\tan(\gamma)} &= \frac{\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}}{\frac{r}{z}} \\ &= \frac{r/x - r/y}{1 + r^2/(xy)} \cdot \frac{z}{r} \\ &= \frac{yz - xz}{xy + r^2}\end{aligned}$$

On the other hand, $\tan\alpha = \frac{r}{x} = \frac{1}{\tan(\beta+\gamma)} \Rightarrow r^2 = \frac{xyz}{x+y+z}$ and $5 = x + y$, $9 = x + z, 7 = x + 7 \Rightarrow x = 3/2,\ y = 7/2,\ z = 11/2 \ldots$.

17. The volume of the pyramid is $\frac{1}{3} \cdot 6^2 \cdot 9 = 108$. The volume of the piece above the plane is $\left(\frac{2}{3}\right)^3 \cdot 108 = 32$, so what is below the plane is....

18. The $\frac{\text{area of } \Delta ABC}{\text{area of } \Delta DEF} = \frac{(AB)^2 \sin A}{(DE)^2 \sin D} = \frac{(2DE)^2 \sin A}{(DE)^2 \sin 2A} = \frac{4}{2\cos A} = \cdots$

19. To draw a reasonable figure, start with an equilateral triangle with horizontal base of length 50. Copy it 4 times, and place the 5 triangles side-by-side with bases alternating between bottom and top. The end result should be an isosceles trapezoid whose bottom base is 15 long and whose parallel top is 100 long. Label the lower left vertex P, the upper left Q, the upper right R, and the lower right S.

    Now keep that figure but add its reflection around the bottom side. On the new bottom of length 100, place 3 more side-by-side equilateral triangles. Label the new lower left vertex U and the lower right T. The triangle bounded by QT, RU, and PS is one of the equilateral triangles...

20. $P(k) = 2^{2^{k+1}}\left(2^{2^{k+1}} + 1\right)$, so $n + (n+1) = 2 \cdot 2^{2^{k+1}} + 1$, which will be greater than 10^{1000}, when $\log\left(2 \cdot 2^{2^{k+1}}\right) > \log 10^{1000}$...