

Hints/Comments on AMATYC Student Math League Exam Spring 2009

1. From one endpoint to the center, $\Delta x = 7 - 3 = 4$ and $\Delta y = 5 - (-3) = 8$, so we should have the same Δx and Δy from the center to the unknown endpoint...
2. $x\Delta(x - 1) = x(x - 1) + (x - 1) = x^2 - 1$. Solving for the positive root of $x^2 - 1 = 323$, we find that $x = 18$...
3. Because the diagonal is $\sqrt{170}$, we know $l^2 + w^2 = 170$. From the perimeter, we also know that $l + w = 18$, and squaring both sides of this equation we find that $l^2 + 2lw + w^2 = 18^2$. Now substituting 170 for $l^2 + w^2$, we get $170 + 2lw = 324$...
4. Any function of the form $f(x) = kx$ satisfies $f(a + b) = f(a) + f(b)$, so such functions will satisfy $f(x + f(x)) = f(x) + f(f(x))$. However, $x = 1$ for the natural logarithm function will make the left side 0 and the right side undefined...
5. The discriminant of the given quadratic equation is $14 + 28k$, and there will be 2 real roots when the discriminant is positive...
6. Factoring both sides, we have $x^{n-2}(x^2 - x - 1) = 7^2 \cdot 41$
7. If $\frac{3}{7}W = \frac{1}{2}M$, then $\frac{W}{M} = \frac{7}{6}$, so there are $7n$ women, $6n$ men, with $3n$ women matched with $3n$ men, for $3n + 3n$ players matched against someone of the opposite gender (out of a total of $7n + 6n$ players)....
8. If the diagonals contain the center, then both are diameters and hence the diagonals are congruent. The quadrilaterals whose diagonals are congruent must be rectangles....
9. Because the central two boxes are each connected to 6 other boxes (and thus each needs to connect to 6 values that are not consecutive to it), those central values must be the 1 and 8. The far left and far right must be the 2 and 7 to avoid being adjacent to the 1 and 8 respectively.

Because 3 and 4 cannot be adjacent, one must be top, the other bottom. Let's assume 3 is top (and flipping the picture around a horizontal line shows that putting 3 on the bottom is really equivalent). The 3 must connect to the 7 (and not the 2). The 4 must be in the other square connected to both 7 and 1, because the 6 can't be there connected to the 7, and the 5 can't be there or it would be next to the 4, which we already said had to be bottom (when we assumed 3 was top). Now 5 goes in the remaining top box, 6 in the remaining bottom box ...
10. Scaling the dimensions of a solid by a factor k scales the volumes by a factor of k^3 , or equivalently, scaling a volume by a factor r scales the lengths by $\sqrt[3]{r}$. Because the volume is scaled by $\frac{1}{2}$, the radius is scaled by $\sqrt[3]{1/2}$
11. Not counting the rung Elena stands on, if there are twice the number above as below, she is $\frac{1}{3}$ of the way up. If moving 5 rungs gets her halfway up, then 5 rungs represents the difference from $\frac{1}{3}$ and $\frac{1}{2}$, from which we find that there are 30 rungs not including the one Elena stands on. If the number of rungs below is 4 times the number above, $4x + x = 30$, and she has 6 rungs above her...

12. $\sin^3\theta - \cos^3\theta = (\sin\theta - \cos\theta)(\sin^2\theta + \sin\theta\cos\theta + \cos^2\theta) = (0.2)(1 + \frac{1}{2}\sin 2\theta)\dots$
13. There 2 vertical asymptotes (when the denominator is zero), and there are 2 horizontal asymptotes: one for $x \rightarrow \infty$ and one for $x \rightarrow -\infty\dots$
14. $(x + 2)^2 + 2 = y^2$, or $y^2 - (x + 2)^2 = 2$. But the difference of two squares is never 2 ...
15. No solution.
16. One strategy is to consider cases, one for each possible number of red lights. There is exactly one way to have 3 red lights (RYRYR). There are 13 ways for 2 red lights: 2 are RYxRY, 2 are RYRYx, 2 are xRYRY, 3 are RYxxR, and 4 are xRYxR. Similarly, there are 29 ways for 1 red, and 11 ways for no reds...
17. $y(6x^2 + y^2 + 10x) = 0$ and $2y(x^2 + x) = 0$. When $y \neq 0$, the second equation gives $x = 0$ or $x = -1$, but we eliminate $x = 0$ because the first equation forces $y = 0$. But then $x = -1 \Rightarrow y^2 = 4\dots$
18. Combining terms on one side and factoring, $0 = (x + y)(x^2 + xy + y^2 - 1)$. The graph of $x + y = 0$ is a line. The graph of the quadratic equation $x^2 + xy + y^2 - 1 = 0$ is a conic section. The discriminant for $x^2 + xy + y^2 - 1 = 0$ is $1^2 - 4(1)(1) < 0$, so its graph is... (Or notice that the equation is not satisfied for large enough values of $|x|$ or $|y|$, so the graph could not be a parabola or hyperbola...)
19. The number $N = 37q$, and for the digits to sum to 16, they must be $\{1, 5, 5, 5, \}$ or $\{1, 1, 5, 9\}$. The first choice does not give a multiple of 37, but $1591 = 37 \cdot 43\dots$
20. The area indicated is the area of $\triangle ABC$ - (sum of areas of two small triangles). The small triangle that includes the leg \overline{AD} has area $\frac{1}{2}(2)(\frac{2}{5} \cdot 8)$, and the small triangle that includes the leg \overline{EC} has area $\frac{1}{2}(\frac{3}{8} \cdot 5)(3)\dots$