Hints/Comments on AMATYC Student Math League Exam Spring 2007

1. If $f(x) = 3x - 1$ and $g(x) = x^2$, find $g(f(f(1)))$: Straightforward algebra problem involving function notation.

2. If we use n for the number of teachers after George buys the new car, then there used to be $n-1$ cars, and the first sentence in the problem gives an equation involving the unknown total number of miles, $n-1$, and 3900. The next sentence gives an equation involving the same unknown total of miles, n, and 36400.

3. Rewrite each $\log x^n$ as $n \log x$.

4. Because we have a fixed mean of 20, a large value of the integers must be "compensated" by small values. The smallest that the other 6 distinct positive integers can be (and still maintain the median of 20) are 1, 2, 3, 20, 21, 22, and 22.

5. The facts that AM and AT share no common factors but their quotient can be written (exactly) as a two-digit decimal less than 1 implies that AT must divide 100. Because AT must be greater than AM, AT cannot be 10, 20, or 50. The remaining two-digit divisor of 100 is 25.

6. We can draw an appropriate diagram and notice that the more stamps exposing edges of length 2, the larger the perimeter.

7. Use the definition of the natural log and properties of exponents to write the product st as a power of e. Then rewrite $e^{5.4}$ as a power of st.

8. Symmetric to the origin $\Rightarrow f(-x) = -f(x)$, and periodic of period 8 $\Rightarrow f(x \pm 8) = f(x)$. Then $f(-4) = -f(4)$ and also $f(-4) = f(-4+8) \Rightarrow f(4) = 0$. Also $f(2) = 3 \Rightarrow f(-6) = 3 \Rightarrow f(6) = -3$

9. Combine the two equations to a single quadratic equation in x, and set the discriminant to be negative.

10. Sketch a figure of a square with $2\sqrt{2}$-wide alleys around the diagonals. The 4 pieces not within the alleys are isosceles triangles of base $6\sqrt{2}$.

11. Because f is increasing, we know that it can intersect its inverse only on the line $y = x$.

12. Draw a right triangle for the acute angle $\theta = \arctan x$. $\cos\theta = x \Rightarrow \frac{1}{\sqrt{1+x^2}} = x$

13. If the bottle holds x gallons, then $\frac{10-x}{10}$ is the fraction of the contents of the jug that is antifreeze after the first replacement, and $10 - \left(\frac{10-x}{10}\right)x$ is the amount of antifreeze in the jug after the second dipping. This amount should be equal to 5.

14. There are only 8 distinct letters in "mathematics", giving $8 \cdot 7 \cdot 6$ possibilities using distinct letters. There will be $7 \cdot 3$ more possibilities with two M's, another $7 \cdot 3$ for 2 A's, and another with 2 T's.

15. $A_{n+1} = \frac{3}{4}A_n + 1500$

16. Place the triangle with right angle at the origin and its two legs on the positive x- and y-axes. The medians of any triangle intersect 2/3 the way from vertex to opposite side.

17. $f(x) = \frac{(x-4)(x+1)}{x+1} \Rightarrow f^{-1}(x) = x+4,\ x \neq -5 \Rightarrow f^{-1}(x) = \frac{(x+4)(x+5)}{x+5}$

18. A reduced form of the coefficient matrix is $\begin{bmatrix} 1 & 1 & k & 1 \\ 0 & -3 & 3 & 1 \\ 0 & 0 & 2-k-k^2 & -2-k \end{bmatrix} \Rightarrow k = -2$

The RREF becomes $\begin{bmatrix} 1 & 0 & -1 & 4/3 \\ 0 & 1 & -1 & -1/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

19. The pentagon is not regular, but the area $= \sum_{i=1}^{5} 2\left(\frac{1}{2} \cdot 3 \cdot x_i\right)$, where x_i is the distance from vertex to point of tangency.

20. Take the tangent function of both sides of the equation, first noting that the equation cannot have -2 as a solution.