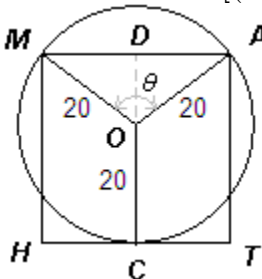


Hints/Comments on AMATYC Student Math League Exam Spring 2008

- $g(2) = g(3 - 1) = 3^2 + 1$
- Each runway has two numbers because there are two angles (separated by 180°) for each runway. The runway with heading 223° also has $223^\circ - 180^\circ = 43^\circ$ as a heading.
- Because $13^3 = 2197$, all three of a , b , and c are no greater than 12.
 $4^3 + 6^3 + 12^3 = 2008$
- If $x^2 + bx + 16 = (x + x_1)(x + x_2)$, then $x_1x_2 = 16$ and $x_1 + x_2 = b$. 16 can be factored as (1)(16) or (2)(8) or (4)(4) or (-1)(-16) or (-2)(-8) or (-4)(-4).
- If $f(x) = x^2 - 2x + 4 = (x - 1)^2 + 3$, then
 $f(x) - f(2y) = (x - 1)^2 + 3 - [(2y - 1)^2 + 3]$
 $= (x - 1)^2 - (2y - 1)^2$ Difference of squares!
 $= [(x - 1) + (2y - 1)][(x - 1) - (2y - 1)]$

6.  Let s represent the length of each side of the square. Let O

be the center of the circle, and let C and D be the midpoints of sides \overline{TH} and \overline{MA} respectively. Using the law of cosines on $\triangle AOM$, we get that

$$\begin{aligned} (MA)^2 = s^2 &= 20^2 + 20^2 - 2 \cdot 20 \cdot 20 \cos \theta \\ &= 2 \cdot 20^2 [1 - \cos \theta] \end{aligned} \quad (*)$$

Because $\cos \theta = \cos\left(2 \cdot \frac{\theta}{2}\right) = 2\cos^2 \frac{\theta}{2} - 1$, we can write

$$\begin{aligned} s^2 &= 2 \cdot 20^2 \left[1 - \left(2\cos^2 \frac{\theta}{2} - 1\right)\right] \\ &= 4 \cdot 20^2 \left[1 - \cos^2 \frac{\theta}{2}\right] \end{aligned} \quad (**)$$

But also the segment from C to D has length s , so

$$s = CD = 20 + 20 \cos \frac{\theta}{2} \quad (***)$$

by looking at the right triangle $\triangle AOD$. We can solve equation (***) for $\cos \frac{\theta}{2}$ in terms of s , then substitute that result in equation (**) to obtain

$$s^2 = 4 \cdot 20^2 \left[1 - \left(\frac{s-20}{20} \right)^2 \right]^2$$

$$s^2 = 4[20^2 - (s-20)^2]^2$$

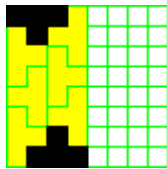
We solve this quadratic equation and reject the extraneous solution $s = 0$.

7. The coin must land on A twice and on M exactly once. The probability of that happening is $\frac{3}{8}$. The die must land on T, Y, and C each exactly once. The probability of that event is $6 \cdot \frac{1}{3^3} = \frac{2}{9}$. So the probability of both happening is $\frac{3}{8} \cdot \frac{2}{9}$.
8. $\frac{\ln 625}{\ln 624} \cdot \frac{\ln 624}{\ln 623} \cdot \frac{\ln 623}{\ln 622} \cdots \frac{\ln 7}{\ln 6} \cdot \frac{\ln 6}{\ln 5} = \frac{\ln 625}{\ln 5} = \frac{\ln 5^4}{\ln 5} =$
9. AMATYC would be written 16 times completely, plus AMAT to fill in 100 squares. So there are 34 A's out of the 100 squares. The chances of getting 3 A's is $\binom{34}{100} \binom{33}{99} \binom{32}{98}$.
10. There are $\frac{5!}{3!2!} = 10$ ways to choose 2 seniors for the committee. So to get at least 600 different committees, there must be at least 60 ways to choose the 3 juniors. $\binom{8}{3} = 56$ and $\binom{9}{3} = 84$...
11. If his drive is D miles and his normal speed is S miles per hour, than his normal trip takes $\frac{D}{S}$ hours, and his trip taking 67% more time than usual takes $1.67\left(\frac{D}{S}\right)$. So

$$\frac{D}{2S} + \frac{D}{2(S+8)} + \frac{D}{(S+8)+6} = 1.67\left(\frac{D}{S}\right)$$

$$\frac{1}{2S} + \frac{1}{2(S+8)} + \frac{1}{(S+8)+6} = \frac{1.67}{S}$$

12. The strategy to have a large number of bags is to take as many 12-pound bags as possible. 83 12-lb bags would weight 996 pounds, which is off from exactly 1000 lbs by 4 pounds. 82 12-lb bags would be off by 16 pounds from 1000 lbs, which still could not be accommodated exactly using 18-lb and 22-lb bats. 81 12-lb bags would be 28 lbs off from 1000, and 80 12-lb bags would be off by 40 lbs, which could be met by 1 18-lb and 1 22 lb for a total of $80 + 1 + 1$ bags.



13. To accommodate the left edge, we could lay 3 of the tiles vertically and overlapping as shown, requiring one tile to be horizontal so that all 8 squares on the left are covered. However, another horizontal tile must be adjacent to the first "column" of tiles, so at least 2 horizontal tiles are needed in the left half of the checkerboard. Similarly, at least 2 will be required for the right, and the pattern shown does cover the left half the board with exactly 2 horizontal tiles.
14. The first biprimes are 6, 10, 14, 15, 21, 22, 26, 33, 34, 35. The largest prime factor of the last three biprimes listed is ...
15. Let P be the property that "the number of counters whose right-hand neighbor is the same color equals the number of counters whose right-hand neighbor is the other

color.” If we have an arrangement of the red and green counters that satisfies P, and if that arrangement has two adjacent red counters, then we could insert a green counter between those adjacent red counters (decreasing the number of same-color right-hand neighbors by one and increasing the number of other-color right-neighbors by two) and adjust by adding 3 more green counters next to any existing green counters. Therefore, any arrangement with the maximum number of green counters will not include any adjacent red counters. So begin with GRGR...RG alternating counters including all eight red and therefore 9 green counters. Each of the eight red and the first eight green counters have an other-color right-neighbor. We need sixteen more green counters to be inserted (anywhere) adjacent to existing green counters so that we satisfy P.

16. $1.5 < \frac{b}{11} < 1.8 \Rightarrow 16.5 < b < 19.8$ or $17 \leq b \leq 19$

$1.5 < \frac{c}{15} < 1.8 \Rightarrow 22.5 < c < 27$ or $23 \leq c \leq 26$

But $\frac{c}{b} \geq 1.5$, and the largest we can get for $\frac{c}{b}$ is when $c = 26$, $b = 17$, and $\frac{26}{17}$ is the only possibility of $\frac{c}{b}$ which is large enough.

17. If we subtract the equations in the system we get

$$\begin{aligned}rs + t - (r + st) &= 24 - 24 \\rs - r + t - st &= 0 \\r(s - 1) - t(s - 1) &= 0 \\(r - t)(s - 1) &= 0\end{aligned}$$

If $r = t$, then the equations in the system are both $rs + r = 24$ or $r(s + 1) = 24$.

The nonnegative solutions become $(1, 23, 1)$, $(2, 11, 2)$, $(3, 7, 3)$, $(4, 5, 4)$, $(6, 3, 6)$, $(8, 2, 8)$, $(12, 1, 12)$. Two of these satisfy $r + s + t = 25$.

If $s = 1$, then both equations of the system become $r + t = 24$, and the nonnegative solutions are $(0, 1, 24)$, $(1, 1, 23)$, $(2, 1, 22)$, \dots , $(12, 1, 12)$, \dots , $(24, 1, 0)$. All 25 of these satisfy $r + s + t = 25$, but we counted $(12, 1, 12)$ already.

18. P is the center of the inscribed circle (and the common distance r is the radius of that circle.) Drawing the radii from P to the nearest point on each side and the segments from P to the points A , B , and C will divide $\triangle ABC$ into 3 pairs of congruent right triangles, with each of the three radii as one of the legs of each of those right triangles. Thus the area of $\triangle ABC$ is the sum of the areas of the six right triangles, and if s is half the perimeter of $\triangle ABC$, we find that the area is rs . On the other hand, Heron's formula gives the area of $\triangle ABC$ as

$\sqrt{s(s - 25)(s - 25)(s - 14)}$. We know that $s = \frac{25+25+14}{2}$, so we can solve for r .

19. The perfect squares 169, 196, 256, 289, 361, 529, 576, 625, 729, 784, 841, and 961 do not have repeated digits. The only perfect square with the digit 3 is 361, and removing all the squares that also have a 6 or 1, we remove 841, so 784 is the only remaining square with a 4. We have 361, 784, and, removing the squares with 7 or 8, we have left 529. $361 + 784 + 529 = 1674$

20. Here's one possible solution: It can be established by direct computation that the sequence is 1, 1, 1, 2, 3, 8, 15, 48, 105, \dots . After the first 2, the terms alternate between odd and even. In fact, the odd terms are $1, 3 = 3 \cdot 1, 15 = 5 \cdot 3 \cdot 1,$

$105 = 7 \cdot 5 \cdot 3 \cdot 1$, $945 = 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1$, \dots , and the even terms are 2 , $8 = 4 \cdot 2$,
 $48 = 6 \cdot 4 \cdot 2$, $384 = 8 \cdot 6 \cdot 4 \cdot 2$, \dots

So for $n \geq 2$, $a_n \cdot a_{n+1} = n!$, and in particular $a_{100}a_{101} = 100!$ The largest power of 5 that is a factor of $100!$ is $5^{100/5 + 100/25}$, or 5^{24} .

(This problem was evidently based on a sequence that appeared in Problem A3 of the 2004 Putnam Exam—that's why it was so challenging!).