Integration Worksheet 2 – Using the Definite Integral

Show all work on your paper as described in class. Video links are included throughout for instruction on how to do the various types of problems. **Important:** Work the problems to match everything that was shown in the videos. For example: Suppose a video shows 3 ways to do a problem, (such as algebraically, graphically, and numerically), then your work should show these 3 ways also. That is, each video is a model for the work I want to see on your paper.

1–R: Read and take notes on Section 8.1

**ESSAY.**

1) **Finding Areas by Slicing**

   Watch and take notes on these videos to see the methods and terminology I want you to use:

   - [http://youtu.be/CqrXaTbx5OQ](http://youtu.be/CqrXaTbx5OQ)
   - [http://youtu.be/DwPM3suqft0](http://youtu.be/DwPM3suqft0)

   There are good hints and notes in the answers to the following, but struggle first before peeking at those!

   (Worksheet items 2 – 9 below have selected exercises from Section 8.1)

2) (a) Write a Riemann sum approximating the area of the region in the figure, using vertical strips as shown.
   (b) Evaluate the corresponding integral. Follow the details as shown in the above videos.

3) (a) Write a Riemann sum approximating the area of the region in the figure, using vertical strips as shown.
   (b) Evaluate the corresponding integral. Follow the details as shown in the above videos.
4) (a) Write a Riemann sum approximating the area of the region in the figure, using horizontal strips as shown.
(b) Evaluate the corresponding integral. Follow the details as shown in the above videos.

5) (a) Write a Riemann sum approximating the area of the region in the figure, using horizontal strips as shown.
(b) Evaluate the corresponding integral. Follow the details as shown in the above videos.
6) **Do 5,6.** Write a Riemann sum and then a definite integral representing the area of the region, using the strip shown. Evaluate the integral exactly.

5.

![Diagram of a rectangle with base 5 and height 3, with a delta x labeled.]  

6.

![Diagram of a triangle with base 6 and height 3, with a delta x labeled.]
7) **Do 7.8.** Write a Riemann sum and then a definite integral representing the area of the region, using the strip shown. Evaluate the integral exactly.

7.

8.
8) **Do 9, 11.** Write a Riemann sum and then a definite integral representing the area of the region, using the strip shown. Evaluate the integral exactly.

9.  

10.  

11.
9) Write a Riemann sum and then a definite integral representing the area of the region, using the strip shown. Evaluate the integral exactly.

MULTIPLE CHOICE. Choose the one alternative that best completes the statement or answers the question.

Find the area of the shaded region.  http://youtu.be/VdTjfRm0oOc

10) \( f(x) = x^3 + x^2 - 6x \)

\[ A) \frac{343}{12} \quad B) \frac{81}{12} \quad C) \frac{937}{12} \quad D) \frac{768}{12} \]
11) \[ y = x^2 - 4x + 3 \]

\[ y = x - 1 \]

A) \( \frac{41}{6} \)  
B) 3  
C) \( \frac{9}{2} \)  
D) \( \frac{25}{6} \)

12) \[ y = x^2 - 2x \]

A) 2  
B) \( \frac{76}{15} \)  
C) \( \frac{7}{15} \)  
D) \( \frac{22}{15} \)
13) \[ y = 2x^2 + x - 6 \quad y = x^2 - 4 \]

A) \( \frac{11}{6} \)  
B) \( \frac{9}{2} \)  
C) \( \frac{19}{3} \)  
D) \( \frac{8}{3} \)

14) \[ y = x - 4 \quad y = \sqrt{2x} \]

A) \( \frac{32}{3} \)  
B) \( \frac{64}{3} \)  
C) 32  
D) \( \frac{128}{3} \)

15) \[ y = 2 \quad y = 2 \sin(\pi x) \]

A) 8  
B) \( \frac{4}{\pi} \)  
C) \( 4 + \frac{4}{\pi} \)  
D) 4
Find the area enclosed by the given curves.

16) \( y = 2x - x^2, \ y = 2x - 4 \)
   A) \( \frac{34}{3} \)  B) \( \frac{31}{3} \)  C) \( \frac{37}{3} \)  D) \( \frac{32}{3} \)

17) \( y = x, \ y = x^2 \)
   A) \( \frac{1}{3} \)  B) \( \frac{1}{2} \)  C) \( \frac{1}{6} \)  D) \( \frac{1}{12} \)

18) Find the area of the region in the first quadrant bounded by the line \( y = 8x \), the line \( x = 1 \), the curve \( y = \frac{1}{\sqrt{x}} \), and the \( x \)-axis.  
   \[ \text{http://youtu.be/ga2urucKKHU} \]
   A) \( \frac{5}{4} \)  B) \( \frac{3}{2} \)  C) 6  D) \( \frac{3}{4} \)

19) Find the area of the region in the first quadrant bounded on the left by the \( y \)-axis, below by the line \( y = \frac{1}{3}x \), above left by \( y = x + 4 \), and above right by \( y = -x^2 + 10 \).
   A) \( \frac{73}{6} \)  B) \( \frac{39}{2} \)  C) 15  D) \( \frac{39}{4} \)

\[ \text{Volumes of solids – VERY IMPORTANT!} \]

ESSAY.

20) Watch and take notes on this video. (It shows an example of finding the volume of a special solid created by revolving a region about a line). I promised at the beginning of the course that you would be able to do this, and you will, very soon!  
   \[ \text{http://youtu.be/po4KO_ds-S4} \]

21) Go to \[ \text{http://demonstrations.wolfram.com/SlicingACylinder/} \] and manipulate this applet by working with the sliders to see how a solid (a cylinder in this case) is cut into "slabs". The idea is that each slab volume would be calculated and then all these would be added up to get the total volume of the cylinder. Copy the figure onto your paper showing the cuts.

22) Watch and take notes on these videos. They will help you with the next section of problems.  

(Worksheet items 23 – 27 below contain selected exercises from Section 8.1)
23) Write a Riemann sum and then a definite integral representing the volume of the region, using the slice shown.

![Diagram of a region with dimensions 9 cm and 4 cm, and variable x and Δx.]

24) Do 14, 15. Write a Riemann sum and then a definite integral representing the volume of the region, using the slice shown.

14.)

![Diagram of a cone with dimensions 6 cm and 4 cm, and variable x and Δx.]

15.)

![Diagram of a cone with dimensions 4 cm and 5 cm, and variable y and Δy.]

25) **Do 16, 17, 18** Write a Riemann sum and then a definite integral representing the volume of the region, using the slice shown.

16.

17.

18.
26) A dam has a rectangular base 1400 meters long and 160 meters wide. Its cross-section is shown in the figure. (The Grand Coulee Dam in Washington state is roughly this size.) By slicing horizontally, set up and evaluate a definite integral giving the volume of material used to build this dam.

![Diagram of a dam with dimensions 1400 m x 160 m x 150 m]  

27) The figure shows a solid with both rectangular and triangular cross sections.  
(a) Slice the solid parallel to the triangular faces. Sketch one slice and calculate its volume in terms of \( x \), the distance of the slice from one end. Then write and evaluate an integral giving the volume of the solid.  
(b) Repeat part (a) for horizontal slices. Instead of \( x \), use \( h \), the distance of a slice from the top.

![Diagram of a solid with rectangular and triangular cross sections]  

Volumes of Solids of Revolution

28–R: Read and take notes on Section 8.2

28) Watch and take notes on these video illustrations (draw the pictures) that show how regions are rotated about a line to create a solid, and then cutting up the solid into disks, cuts, slabs, rings, slabs with holes etc. to find the volume of the solid.

29) Go to [http://demonstrations.wolfram.com/SolidsOfRevolution/](http://demonstrations.wolfram.com/SolidsOfRevolution/) and move the angle slider down to rotate the region which then creates the solid of revolution. Copy the graphs to your paper.

30) Go to [http://tutorial.math.lamar.edu/Classes/CalcII/VolumeWithRings.aspx](http://tutorial.math.lamar.edu/Classes/CalcII/VolumeWithRings.aspx) and copy the region and resulting 3-D rotations from example 1, 2, 3, 4 onto your paper. Take notes.

31) Go to [http://mathdemos.org/mathdemos/diskmethod/diskmethodgallery.html](http://mathdemos.org/mathdemos/diskmethod/diskmethodgallery.html) and click on the gifs to show the animation of the cuts of the solids. Copy the pictures to your paper.


(Worksheet items 33 – 37, 39 below contain selected problems from Section 8.2)

33) a) The region in the figure is rotated around the x-axis. Using the strip shown, write an integral giving the volume.
   b) Evaluate the integral.

![Diagram](https://via.placeholder.com/150)

34) (a) The region in the figure is rotated around the y-axis. Write an integral giving the volume.
   (b) Evaluate the integral.

![Diagram](https://via.placeholder.com/150)
35) **Do 7-13 odd:** The region described is rotated around the x-axis. Find the volume.

7. Bounded by \( y = 4 - x^2, y = 0, x = -2, x = 0. \)
8. Bounded by \( y = \sqrt{x + 1}, y = 0, x = -1, x = 1. \)
9. Bounded by \( y = e^x, y = 0, x = -1, x = 1. \)
10. Bounded by \( y = \cos x, y = 0, x = 0, x = \pi/2. \)
11. Bounded by \( y = 1/(x + 1), y = 0, x = 0, x = 1. \)
12. Bounded by \( y = \sqrt{\cosh 2x}, y = 0, x = 0, x = 1. \)
13. Bounded by \( y = x^2, y = x, x = 0, x = 1. \)

(in case you are wondering about #12, hyperbolic cosine (cosh) is covered in Calculus 2)

36) **Do 25, 27:** Set up, but do not evaluate, an integral that represents the volume obtained when the region in the first quadrant is rotated about the given axis.

25. Bounded by \( y = 3\sqrt{x}, x = 4y. \) Axis \( x = 9. \)
26. Bounded by \( y = 3\sqrt{x}, x = 4y. \) Axis \( y = 3. \)
27. Bounded by \( y = 0, x = 9, y = \frac{1}{3}x. \) Axis \( y = -2. \)

37) **Do 37, 39:** Sketch the solid obtained by rotating each region around the indicated axis. Using the sketch, show how to approximate the volume of the solid by a Riemann sum, and hence find the volume.

37. Bounded by \( y = x^3, x = 1, y = -1. \) Axis: \( y = -1. \)
38. Bounded by \( y = \sqrt{x}, x = 1, y = 0. \) Axis: \( x = 1. \)
39. Bounded by the first arch of \( y = \sin x, y = 0. \) Axis: \( x \)

38) **Go to** [http://demonstrations.wolfram.com/SolidsOfKnownCrossSection/](http://demonstrations.wolfram.com/SolidsOfKnownCrossSection/) and copy down the pictures for each of the solids of known cross section. These are NOT solid of revolution, they are solids created by known (square, semicircle, triangle) cross sections. These types of 3-D images are hard to imagine so this demonstration helps you see them. **Watch and take notes on this video:** [http://youtu.be/wSD-egbSujs](http://youtu.be/wSD-egbSujs)
39) **Do 41, 43, 45.** These concern the region bounded by \( y = x^2, \ y = 1, \) and the \( y \)-axis, for \( x \geq 0. \) Find the volume of the following solids.

41. The solid obtained by rotating the region about the \( x \)-axis.

42. The solid obtained by rotating the region about the line \( y = -2. \)

43. The solid whose base is the region and whose cross-sections perpendicular to the \( x \)-axis are squares.

44. The solid whose base is the region and whose cross-sections perpendicular to the \( x \)-axis are semicircles.

45. The solid whose base is the region and whose cross-sections perpendicular to the \( y \)-axis are equilateral triangles.

**MULTIPLE CHOICE.** Choose the one alternative that best completes the statement or answers the question.

Find the volume of the described solid.

40) The solid lies between planes perpendicular to the \( x \)-axis at \( x = 0 \) and \( x = 6. \) The cross sections perpendicular to the \( x \)-axis between these planes are squares whose bases run from the parabola \( y = -3\sqrt{x} \) to the parabola \( y = 3\sqrt{x}. \) [External Video Link]

\[ A) \ 630 \quad B) \ 648 \quad C) \ 108 \quad D) \ 324 \]

41) The solid lies between planes perpendicular to the \( x \)-axis at \( x = -2 \) and \( x = 2. \) The cross sections perpendicular to the \( x \)-axis between these planes are squares whose bases run from the semicircle \( y = -\sqrt{4 - x^2} \) to the semicircle \( y = \sqrt{4 - x^2}. \)

\[ A) \ \frac{64}{3} \quad B) \ \frac{32}{3} \quad C) \ \frac{128}{3} \quad D) \ \frac{16}{3} \]

42) The base of the solid is the disk \( x^2 + y^2 \leq 25. \) The cross sections by planes perpendicular to the \( y \)-axis between \( y = -5 \) and \( y = 5 \) are isosceles right triangles with one leg in the disk.

[External Video Link]

\[ A) \ \frac{1000}{3} \quad B) \ \frac{250}{3} \quad C) \ \frac{1250}{3} \quad D) \ \frac{2000}{3} \]

43) The solid lies between planes perpendicular to the \( x \)-axis at \( x = -2 \) and \( x = 2. \) The cross sections perpendicular to the \( x \)-axis are semicircles whose diameters run from \( y = -\sqrt{4 - x^2} \) to \( y = \sqrt{4 - x^2}. \)

\[ A) \ \frac{16}{3} \pi \quad B) \ \frac{64}{3} \pi \quad C) \ \frac{32}{3} \pi \quad D) \ \frac{8}{3} \pi \]
44) The solid lies between planes perpendicular to the x-axis at \(x = -4\) and \(x = 4\). The cross sections perpendicular to the x-axis are circular disks whose diameters run from the parabola \(y = x^2\) to the parabola \(y = 32 - x^2\).

\[
\begin{align*}
\text{A)} & \quad \frac{256}{3} \pi \\
\text{B)} & \quad \frac{8192}{5} \pi \\
\text{C)} & \quad \frac{16384}{15} \pi \\
\text{D)} & \quad \frac{8192}{15} \pi
\end{align*}
\]

45) The base of a solid is the region between the curve \(y = 3\cos x\) and the x-axis from \(x = 0\) to \(x = \pi/2\). The cross sections perpendicular to the x-axis are squares with bases running from the x-axis to the curve.

\[
\begin{align*}
\text{A)} & \quad 2\pi \\
\text{B)} & \quad \frac{9}{2} \pi \\
\text{C)} & \quad \frac{3}{2} \pi \\
\text{D)} & \quad \frac{9}{4} \pi
\end{align*}
\]

Find the volume of the solid generated by revolving the shaded region about the given axis.

46) About the x-axis

\[
\begin{align*}
\text{A)} & \quad 12\pi \\
\text{B)} & \quad \frac{224}{3} \pi \\
\text{C)} & \quad \frac{32}{3} \pi \\
\text{D)} & \quad \frac{64}{3} \pi
\end{align*}
\]

47) About the x-axis

\[
\begin{align*}
\text{A)} & \quad \frac{1053}{5} \pi \\
\text{B)} & \quad \frac{648}{5} \pi \\
\text{C)} & \quad \frac{3159}{5} \pi \\
\text{D)} & \quad 18\pi
\end{align*}
\]
48) About the y-axis

\[ x = \frac{6y}{7} \]

A) \( 98\pi \)  B) \( 21\pi \)  C) \( 168\pi \)  D) \( 84\pi \)

49) About the y-axis

\[ y = \sqrt{5x} \]

A) \( \frac{25}{3}\pi \)  B) \( 25\pi \)  C) \( 50\pi \)  D) \( 625\pi \)

50) About the y-axis

\[ x = \frac{y^2}{3} \]

A) \( \frac{27}{5}\pi \)  B) \( \frac{108}{5}\pi \)  C) \( \frac{45}{2}\pi \)  D) \( 18\pi \)
51) About the x-axis

\[ y = 4 - x^2 \]

52) About the x-axis

\[ y = 3\sqrt{\sin x} \]

53) Find the volume of the solid generated by revolving the region bounded by the given lines and curves about the x-axis.

\[ y = x, \ y = 0, \ x = 2, \ x = 6 \]

A) 16\pi

B) 20\pi

C) \frac{208}{3}\pi

D) \frac{4}{3}\pi

54) \[ y = x^2, \ y = 0, \ x = 0, \ x = 6 \]

A) \frac{7776}{5}\pi

B) 72\pi

C) 324\pi

D) 1944\pi

55) \[ y = \sqrt{2x + 3}, \ y = 0, \ x = 0, \ x = 1 \]

A) 2\pi

B) \pi

C) 4\pi

D) \frac{3\pi}{2}
56) \( y = \frac{1}{x}, \ y = 0, \ x = 1, \ x = 8 \)

A) \( \frac{7}{16} \pi \)  
B) \( \pi \ln 8 \)  
C) \( \frac{3}{8} \pi \)  
D) \( \frac{7}{8} \pi \)

57) \( y = x^2, \ y = 16, \ x = 0 \)

A) \( \frac{6144}{5} \pi \)  
B) \( \frac{4096}{5} \pi \)  
C) \( \frac{1024}{5} \pi \)  
D) \( \frac{128}{3} \pi \)

Find the volume of the solid generated by revolving the region about the given line.

58) The region bounded above by the line \( y = 16 \), below by the curve \( y = 16 - x^2 \), and on the right by the line \( x = 4 \), about the line \( y = 16 \)

A) \( \frac{7168}{15} \pi \)  
B) \( \frac{64}{3} \pi \)  
C) \( \frac{8192}{15} \pi \)  
D) \( \frac{1024}{5} \pi \)

59) The region in the second quadrant bounded above by the curve \( y = 4 - x^2 \), below by the \( x \)-axis, and on the right by the \( y \)-axis, about the line \( x = 1 \)

A) \( \frac{256}{15} \pi \)  
B) \( \frac{32}{3} \pi \)  
C) \( \frac{56}{3} \pi \)  
D) \( 8\pi \)

Solve the problem.

60) The disk \((x - 6)^2 + y^2 \leq 4\) is revolved about the \( y \)-axis to generate a torus. Find its volume. (Hint: \( \int_{-2}^{2} \sqrt{4 - y^2} \, dy = 2\pi \), since it is the area of a semicircle of radius 2.)  

http://youtu.be/uM55H6rBdy0

A) \( 24\pi^2 \)  
B) \( 48\pi^2 \)  
C) \( 12\pi^2 \)  
D) \( 24\pi \)

61) The hemispherical bowl of radius 5 contains water to a depth 1. Find the volume of water in the bowl.

A) \( \frac{7}{3} \pi \)  
B) \( \frac{14}{3} \pi \)  
C) \( \frac{139}{3} \pi \)  
D) \( 88\pi \)

62) A water tank is formed by revolving the curve \( y = 5x^4 \) about the \( y \)-axis. Find the volume of water in the tank as a function of the water depth, \( y \).

A) \( V(y) = \frac{3\pi}{2\sqrt{5}} y^{3/2} \)  
B) \( V(y) = \frac{2\pi}{3\sqrt{5}} y^{3/2} \)  
C) \( V(y) = \frac{\pi}{2\sqrt{5}} y^{1/2} \)  
D) \( V(y) = \frac{\pi}{9} y^9 \)

63) A right circular cylinder is obtained by revolving the region enclosed by the line \( x = r \), the \( x \)-axis, and the line \( y = h \), about the \( y \)-axis. Find the volume of the cylinder.

A) \( \pi rh \)  
B) \( \pi rh^2 \)  
C) \( 2\pi r^2h \)  
D) \( \pi r^2h \)
64) Find the volume that remains after a hole of radius 1 is bored through the center of a solid sphere of radius 2.

\[ \text{http://youtu.be/HSBG3tkuzTM} \]

A) \( \frac{5}{3} \pi \)

B) \( \frac{8}{3} \pi \)

C) \( \frac{10}{3} \pi \)

D) \( \frac{32}{3} \pi \)

**The Shell Method**, another method for finding the volume of a solid of revolution.

**ESSAY.**

65) Watch and take notes on this video illustrating the **Shell Method**, another method for finding the volume of a solid of revolution. Especially pay attention to the applet he shows at 39 seconds into the video:

\[ \text{http://www.youtube.com/watch?v=NyBX5D1cAMg} \]

AND, watch this gif and take notes:

\[ \text{http://mathdemos.org/mathdemos/shellmethod/gallery/quad1_shellsmovie.gif} \]

**MULTIPLE CHOICE.** Choose the one alternative that best completes the statement or answers the question.

Use the **shell method** to find the volume of the solid generated by revolving the shaded region about the indicated axis.

66) About the y-axis

\[ \text{http://youtu.be/Q377Ts4Wlmw} \]

\[ y = 2x - x^2 \]

A) \( 2 \pi \)

B) \( 4 \pi \)

C) \( \frac{8}{3} \pi \)

D) \( \frac{4}{3} \pi \)
67) About the x-axis  [Link](http://youtu.be/uqrS5CDriGw)

\[ y = \sqrt{1 - x^2} \]

A) \( \frac{2}{3} \pi \)  
B) \( \frac{3}{2} \pi \)  
C) \( \frac{1}{3} \pi \)  
D) \( 1 \pi \)

68) About the y-axis

\[ y = 3 \sin(x^2) \]

A) 12\pi  
B) 6\pi  
C) 9\pi  
D) 3\pi

Use the shell method to find the volume of the solid generated by revolving the region bounded by the given curves and lines about the y-axis.

69) \( y = x^2, \ y = 4 + 3x, \) for \( x \geq 0 \)  [Link](http://youtu.be/TGB0bJtTf_c)

A) 32\pi  
B) 192\pi  
C) 96\pi  
D) 64\pi

70) \( y = 7x, \ y = -\frac{x}{7}, \ x = 1 \)

A) \( \frac{33}{7} \pi \)  
B) \( \frac{100}{21} \pi \)  
C) \( \frac{50}{21} \pi \)  
D) 50\pi

71) \( y = \frac{2}{\sqrt{x}}, \ y = 0, \ x = 1, \ x = 4 \)

A) \( \frac{56}{3} \pi \)  
B) \( \frac{64}{3} \pi \)  
C) 24\pi  
D) \( \frac{28}{3} \pi \)
72) \( y = 4e^{-x^2}, y = 0, x = 0, x = 1 \)
A) \( 8 \left( 1 - \frac{1}{e} \right) \pi \)  
B) \( 4(e - 1) \pi \)  
C) \( 4 \left( 1 - \frac{1}{e} \right) \pi \)  
D) \( 8 \left( 1 + \frac{1}{e} \right) \pi \)

Use the shell method to find the volume of the solid generated by revolving the region bounded by the given curves and lines about the x-axis.

73) \( x = 3\sqrt{y}, x = -3y, y = 1 \)
A) \( \frac{22}{5} \pi \)  
B) \( 6\pi \)  
C) \( 8\pi \)  
D) \( \frac{11}{5} \pi \)

74) \( y = \sqrt{x}, y = 0, y = x - 6 \)
A) \( \frac{225}{2} \pi \)  
B) \( \frac{63}{2} \pi \)  
C) \( 27\pi \)  
D) \( \frac{63}{4} \pi \)

Use the shell method to find the volume of the solid generated by revolving the region bounded by the given curves about the given lines.

75) \( y = 4 - x^2, \quad y = 4, \quad x = 2; \) revolve about the line \( y = 4 \)  
A) \( \frac{256}{15} \pi \)  
B) \( \frac{224}{15} \pi \)  
C) \( \frac{8}{3} \pi \)  
D) \( \frac{32}{5} \pi \)

76) \( y = 2x, \quad y = 0, \quad x = 2; \) revolve about the line \( x = -3 \)
A) \( \frac{52}{3} \pi \)  
B) \( \frac{104}{3} \pi \)  
C) \( -\frac{40}{3} \pi \)  
D) \( \frac{52}{3} \)

**Finding the length of a curve (Arclength)**

**ESSAY.**

77) Watch and take notes on this video:  [http://youtu.be/sZaMtIlQ35w](http://youtu.be/sZaMtIlQ35w)

**MULTIPLE CHOICE.** Choose the one alternative that best completes the statement or answers the question.

Find the length of the curve.

78) \( y = 2x^{3/2} \) from \( x = 0 \) to \( x = \frac{5}{4} \)  
A) \( \frac{9}{4} \)  
B) \( \frac{335}{3} \)  
C) \( \frac{335}{72} \)  
D) \( \frac{335}{108} \)

79) \( y = \frac{1}{6}x^3 + \frac{1}{2x} \) from \( x = 1 \) to \( x = 5 \)
A) \( \frac{316}{15} \)  
B) \( \frac{79}{5} \)  
C) \( \frac{632}{15} \)  
D) \( \frac{127}{6} \)
Set up an integral for the length of the curve.

\[ y = x^4, 0 \leq x \leq 1 \]

A) \[ \int_{0}^{1} \sqrt{1 + 16x^8} \, dx \]

B) \[ \int_{0}^{1} \sqrt{1 + 4x^6} \, dx \]

C) \[ \int_{0}^{1} \sqrt{1 + 4x^3} \, dx \]

D) \[ \int_{0}^{1} \sqrt{1 + 16x^6} \, dx \]

**Work**

**81–R:** Read and take notes on Section 8.5 JUST THE WORK PART!

**ESSAY.**

**81) Write these notes on your paper:**

In physics the word “work” has a technical meaning which is different from its everyday meaning. Physicists say that if a constant force, F, is applied to some object to move it a distance, d, then the force has done work on the object. The force must be parallel to the motion (in the same or the opposite direction). We make the following definition:

**Work done = Force \cdot Distance or W = F \cdot d**

We’ll see two types of WORK problems here: 1) The work needed to lift objects up to a location, or to pump liquid up and out of a container, 2) The work needed to build structures from the ground up.

**82) Watch this video about WORK:** The back story is that Maynard G. Crebs is a "beatnik" (in the 1950's) and therefore stereotypically doesn't like the idea of "work":

[http://www.youtube.com/watch?v=TgecgpCfAYo](http://www.youtube.com/watch?v=TgecgpCfAYo). You may feel the same about WORK after you do the following problems!

**83) An anchor weighing 100 lb in water is attached to a chain weighing 3 lb/ft in water. Find the work done to haul the anchor and chain to the surface of the water from a depth of 25 ft.**


**84) A construction worker is standing on the roof of a 20 ft. above the ground. A 50 lbs tool bag is attached to a rope weighing 4 lbs per foot. Find the work needed to lift the bag from the ground up to the roof.**

**85) A rectangular water tank has length 20 ft., width 10 ft., and depth 15 ft. If the tank is full, how much work does it take to pump all the water out (up and out or the top)? Water weighs 62.4 lbs/ft³**


**86) A water tank is in the form of a right circular cylinder with height 20 ft and radius 6 ft. If the tank is half full of water, find the work required to pump all of it over the top rim. Water weighs 62.4 lbs/ft³**

[http://youtu.be/6ltq41shW5A](http://youtu.be/6ltq41shW5A)

**87) Suppose the tank in the previous problem is full of water. Find the work required to pump all of it to a point 8 ft above the top of the tank. Water weighs 62.4 lbs/ft³**

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88) Water in a cylinder of height 10 ft and radius 4 ft is to be pumped out. Water weighs 62.4 lbs/ft³. Find the work required if
(a) The tank is full of water and the water is to pumped over the top of the tank.
(b) The tank is full of water and the water must be pumped to a height 5 ft above the top of the tank.
(c) The depth of water in the tank is 8 ft and the water must be pumped over the top of the tank.

89) A radius 10 ft, height 100 ft. cone monument (as shown below) is to be constructed with stones weighing 50 lbs/ft³. How much work is required? Hint: You want to first find out how much a slab cut up at a y height weighs and how much work it will take to lift it from the ground up to that y location. So you need the radius of the slab cut. Set up a ratio to get this in terms of y.

90) It is reported that the Great Pyramid of Egypt was built in 20 years. If the stone making up the pyramid has a density of 200 pounds per cubic foot, find the total amount of work done in building the pyramid. The pyramid is 410 feet high and has a square base 755 feet by 755 feet. Hint: Set up a ratio to get the length of the slab cut, s, up at a h height location. (I used s and h here since the answer key does.)

91) In the previous problem about the Great Pyramid of Egypt, you calculated the total work done in building the pyramid. Now estimate the total number of workers needed. Assume every laborer worked 10 hours a day, 300 days a year, for 20 years. Assume that a typical worker lifted ten 50 pound blocks a distance of 4 feet every hour. Doing some calculations and your answer to the previous problem, how many workers were needed?

92) A cylindrical garbage can of depth 3 ft and radius 1 ft fills with rainwater up to a depth of 2 ft. How much work would be done in pumping the water up to the top edge of the can? (Water weighs 62.4 lb/ft³.)

Solve the problem.
93) A 20 ft long chain hangs vertically from a cylinder attached to a winch. Assume there is no friction in the system and that the chain has a density of 5 lbs/ft. How much work is required to wind the entire chain onto the cylinder using the winch?
94) A small pool has the shape of a box with a base that measures 20 ft by 19 ft and a depth of 2 ft. How much work is required to pump the water out of the pool when it is full? (density of water = 62.4 \( \frac{\text{lbs}}{\text{ft}^3} \))

95) A cylindrical water tank has height 10 ft and radius 3 ft (see figure). If the tank is full of water, how much work is required to pump the water to the level of the top of the tank and out of the tank? (density of water = 62.4 \( \frac{\text{lbs}}{\text{ft}^3} \))

[Diagram of a cylindrical water tank with height 10 ft and radius 3 ft]

96) A spherical water tank with an inner radius of 4 ft has its lowest point 3 ft above the ground. It is filled by a pipe that feeds the tank at its lowest point (see figure). Neglecting the volume of the inflow pipe, how much work is required to fill the tank if it is initially empty? Hint: Think of the tank after it is full and consider how much work it took to lift each slab of water in the sphere from the ground up to its position.

(density of water = 62.4 \( \frac{\text{lbs}}{\text{ft}^3} \))

[Diagram of a spherical water tank with inner radius 4 ft and lowest point 3 ft above the ground]
97) A water trough has a semicircular cross section with a radius of 2 ft and a length of 5 ft (see figure). How much work is required to pump water out of the trough when it is full? (density of water = 62.4 lbs/ft³)

98) A glass has circular cross sections that taper (linearly) from a radius of 8 in. at the top of the glass to a radius of 7 in. at the bottom. The glass is 14 in. high and full of lemonade. How much work is required to drink all the lemonade through a straw if your mouth is 6 in. above the top of the glass? Assume the density of lemonade equals the density of water, (density of water = 62.4 lbs/ft³ = 0.0361 lbs/in³).

THE END !!!
1)

2) a) Area \( \approx \sum_{x: 0 \text{ to } 3} 2x \Delta x \) (This is the short-hand sum notation we want. We don’t bother with the i counter, we just want to set up a sum that continually reminds us we are adding up slices.) And, notice the Riemann sum is an approximation so we put "\( \approx \)" or "\( \cong \)", whereas the integral is the exact area because the limit has been taken, so we put the "=" sign. b) Area = \( \int_{0}^{3} 2x \, dx = 9 \) (The last step is to set up the integral and evaluate it. Verify by geometry, you can (as in this one) and verify your by-hand algebra simplifications by running the Integral program.

3) a) Area \( \approx \sum_{x: 0 \text{ to } 6} (-x^2 + 6x) \Delta x \) b) Area = \( \int_{0}^{6} (-x^2 + 6x) \, dx = 36 \)

4) a) Area \( \approx \sum_{y: 0 \text{ to } 6} (3 - y/2) \Delta y \) b) Area = \( \int_{0}^{6} (3 - y/2) \, dy = 9 \) . You always should get the same answer using vertical or horizontal cuts (although one direction may be harder than the other).

5) a) We must solve for x in terms of y to see up at a y location what the left x value and right x value of the strip is. 
\[ y = -x^2 + 6x \] 
\[ \Rightarrow x^2 - 6x + y = 0 \] 
\[ = \Rightarrow x = 3 \pm \sqrt{9 - y} \] (after simplifying from the quadratic formula) \( \Rightarrow \) the length of the strip up at a y location is: \( (3 + \sqrt{9 - y}) - (3 - \sqrt{9 - y}) = 2\sqrt{9 - y} \)

so Area \( \approx \sum_{y: 0 \text{ to } 9} 2\sqrt{9 - y} \Delta y \) b) Area = \( \int_{0}^{9} 2\sqrt{9 - y} \, dy = 36 \) This is the same area answer as was obtained in the previous problem with the same figure, it just turned out to be a much more difficult direction to do the cuts. You always should get the same answer using vertical or horizontal cuts (although one direction may be harder than the other).

6) 5) Area \( \approx \sum_{x: 0 \text{ to } 5} 3 \Delta x \), Area = \( \int_{0}^{5} 3 \, dx = 15 \) 6) To see what the y height value is at an x location, set up the similar triangle ratio: 
\[ \frac{y}{x} = \frac{3}{6} \] 
\[ \Rightarrow y = (1/2)x \] . As x moves right to left (as the picture instructs us to do) we obtain the following sum for the area of the figure: 
Area \( \approx \sum_{x: 0 \text{ to } 6} \frac{1}{2}x \Delta x \), Area = \( \int_{0}^{6} \frac{1}{2}x \, dx = 9 \)
7) Label the strip width \( a \). Then set up a similar triangle ratio: \( \frac{a}{5-h} = \frac{3}{5} \Rightarrow a = \frac{3}{5}(5-h) \). So up at an \( h \) height location the width of the strip is \( \frac{3}{5}(5-h) \). So Area \( \approx \sum_{h:0}^{5} \frac{3}{5}(5-h) \Delta h \). Lets check the extremes before going on: Down on the ground \( h=0 \) so the strip should be 3 wide there: \( \frac{3}{5}(5-0) = 3 \), check! Up at the top \( h=5 \) so the strip should be 0 wide there: \( \frac{3}{5}(5-5) = 0 \), check! So Area \( = \int_{0}^{5} \frac{3}{5}(5-h) \, dx = \frac{15}{2} \) 

8) This one was shown in a video: 

http://www.youtube.com/watch?v=DwPM3suqft0&feature=youtu.be 

The answer is \( 9\pi \) 

9) Up at a \( y \) height location the width of the strip is the \( x \) value solved for in terms of \( y \): \( x = \sqrt{10-y^2} \) 

so Area \( \approx \sum_{y:0}^{\sqrt{10}} \sqrt{10-y^2} \Delta y \), Area \( = \int_{0}^{\sqrt{10}} \sqrt{10-y^2} \, dy = (5/2)\pi \) (exact value, but for this one just run the Integral program, I'll take it!) 

11) You must solve each equation for \( x \) in terms of \( y \) to find the width of the strip up at a \( y \) height location, so Area \( \approx \sum_{y:0}^{1} (y-y^2) \Delta y \), Area \( = \int_{0}^{1} (y-y^2) \, dy = 1/6 \) 

9) The height of the strip at an \( x \) location is \( (6-3x) - (x^2 - 4) \) which simplifies to \( -x^2 - 3x + 10 \) so Area \( \approx \sum_{x:0}^{2} (-x^2 - 3x + 10) \Delta x \), Area \( = \int_{0}^{2} (-x^2 - 3x + 10) \, dx = \frac{34}{3} \) 

10) C 
11) C 
12) C 
13) C 
14) B 
15) D 
16) D 
17) C 
18) A 
19) A 
20) 
21) 
22) 
23) Each slab is a cylinder with volume \( \pi(2)^2 \Delta x \), so Volume \( \approx \sum_{x:0}^{9} 4\pi \Delta x \) cm\(^3\), Volume \( = \int_{0}^{9} 4\pi \, dx = 36 \pi \)
24) 14) We must find the radius of the slab at an x location. Set up ratio: \( \frac{x}{6} = \frac{2}{3} \Rightarrow r = (1/3)x \), Volume of slab = \( \pi \left( \frac{x^2}{9} \right) \triangle x \)

Volume = \( \sum_{x: 0 \text{ to } 6} \frac{\pi}{9} x^2 \triangle x \text{ cm}^3 \), Volume = \( \int_0^6 \frac{\pi}{9} x^2 \, dx = 8\pi \text{ cm}^3 \)

15) answer: Volume = \( \int_0^5 \left( \frac{4\pi}{25} \right) y^2 \, dy = \frac{20\pi}{3} \text{ cm}^3 \)

25) 16) Each slab cut is a rectangular solid. We must find the width of the rectangle side up at a y location. Width = \( \sqrt{49-y^2} \), so slab volume = \( 2\sqrt{49-y^2} (10) \triangle y \), Volume = \( \sum_{y: 0 \text{ to } 7} 20\sqrt{49-y^2} \triangle y \text{ m}^3 \), Volume = \( \int_0^7 20\sqrt{49-y^2} \, dy = 245\pi \text{ m}^3 \) (For this one I would except you running the Integral program to get a decimal approximation.)

17) answer:

\[ \int_0^5 \pi (5^2 - y^2) \, dy = \frac{250\pi}{3} \text{ mm}^3 \]

18) Up at a y location we find the width of the slab: \( \frac{a}{2-y} = \frac{2}{2} \Rightarrow a = 2-y \), so the volume of the slab = \( (2-y)(2-y) \triangle y \), Volume = \( \sum_{y: 0 \text{ to } 2} (2-y)^2 \triangle y \), Volume = \( \int_0^2 (2-y)^2 \, dy = \int_0^2 (y-2)^2 \, dy = \frac{(y-2)^3}{3} \) evaluated from 0 to 2 = 0

- \( -\frac{8}{3} = \frac{8}{3} \text{ m}^3 \) (notice the trick of turning the subtraction around when the quantity is an even power.) Notice also that the volume of a pyramid is \( (1/3) \text{ (area base) (height)} = (1/3)(4)(2) = \frac{8}{3} \text{ m}^3 \) so the calculus answer checks out.

26) answer:

\[ \int_0^{150} 1400(160 - h) \, dh = 1.785 \cdot 10^7 \text{ m}^3 \]
27) answer:
(a) \(3 \Delta x\);
\[\int_0^4 3 \, dx = 12 \text{ cm}^3\]

(b) \(8(1 - h/3) \Delta h\);
\[\int_0^3 8(1 - h/3) \, dh = 12 \text{ cm}^3\]

33) Draw the picture, then draw a representative cut (a disk). At an \(x\) location this disk has radius \(2x\) so the volume of the disk is \(\pi(2x)^2 \Delta x\). Develop the sum and then the integral from there (\(x: 0 \text{ to } 3\))

(a) \(4\pi \int_0^3 x^2 \, dx\)
(b) \(36\pi\)

(a) \(\pi/4 \int_0^6 (36 - y^2) \, dy\)
(b) \(36\pi\)
35) answers:
   7 \( \frac{256\pi}{15} \)
   9 \( \pi(\frac{e^2}{2} - \frac{e^{-2}}{2}) \)
   11 \( \frac{\pi}{2} \)
   13 \( \frac{2\pi}{15} \)

36) answers:
   25 \( V = \int_0^2 \left[ \pi(9 - y^3)^2 - \pi(9 - 4y)^2 \right] dy \)
   27 \( V = \int_0^9 \left[ \pi(2 + \frac{1}{3}x)^2 - \pi 2^2 \right] dx \)

37) answers:
   37 \( V = (\frac{16}{7})\pi \approx 7.18 \)

39 \( V = (\frac{\pi^2}{2}) \approx 4.935 \)

38) 39) answers:
   41 \( 4\pi/5 \)
   43 \( 8/15 \)
   45 \( \sqrt{3}/8 \)
40) B
41) C
42) A
43) A
44) C
45) D
46) C
47) B
48) D
49) B
50) B
51) C
52) A
53) C
54) A
55) C
56) D
57) B
58) D
59) C
60) B
61) B
62) B
63) D
64) C
65) C
66) C
67) A
68) B
69) D
70) B
71) A
72) C
73) A
74) B
75) D
76) B
77)
78) D
79) A
80) D
81)
82)
83) see video
84) \( (50)(20) + \int_{0}^{20} 4(20-y) \, dy = 1000 + 800 = 1800 \text{ ft-lbs.} \)
85) see the video
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86) see the video

87) Work \approx \sum_{y: 0 \text{ to } 20} \pi 6^2 \Delta y \cdot 62.4 \cdot (28-y) , \quad \text{Work} = \int_0^{20} 36\pi(62.4)(28-y) \, dy = 2540618.5 \text{ ft-lbs}

88) (a) 156,828 ft-lbs (b) 313,656 ft-lbs (c) 150,555 ft-lbs

89) \frac{a}{100-y} = \frac{10}{100} \Rightarrow \text{get } a \text{ in terms of } y. \quad \text{Then, the weight of the slab } = \pi \left(\frac{1}{10}(100-y)\right)^2 \Delta y \text{ ft}^3 \cdot 50 \frac{\text{lbs}}{\text{ft}^3}

So the work needed to lift the slab from the ground up to the y location is:

\pi \left(\frac{1}{10}(100-y)\right)^2 \Delta y \text{ ft}^3 \cdot 50 \frac{\text{lbs}}{\text{ft}^3} \cdot y \text{ ft.} = \pi \left(\frac{1}{10}(100-y)\right)^2 \Delta y \cdot 50 \cdot y \text{ ft-lbs}

\text{Work} = \int_0^{100} \pi \left(\frac{1}{10}(100-y)\right)^2 \cdot 50 \cdot y \, dy = 13089969.4 \text{ ft-lbs}

90) Answer:

91) answer:

92) 784.14 ft-lb
93) 1000 ft-lbs
94) 47424 ft-lbs
95) 88216 ft-lbs
96) 117098 ft-lbs
97) 1663.8 ft-lbs
98) 94.6 ft-lbs